

TOTAL MEAN CURVATURE OF IMMERSED SURFACES IN E^m

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ABSTRACT. Total mean curvature and value-distribution of mean curvature for certain pseudo-umbilical surfaces are studied.

1. Introduction. In the classical theory of surfaces in a euclidean m -space E^m , the two most important curvatures are the so-called Gauss curvature G and the mean curvature α . It is well known that the Gauss curvature is intrinsic. The integral of Gauss curvature gives the beautiful Gauss-Bonnet formula, which holds for orientable compact surfaces as well as nonorientable ones,

$$(1.1) \quad \int_M G dV = 2\pi\chi(M),$$

where dV and $\chi(M)$ denote the volume element and Euler-Poincaré characteristic of M . For the mean curvature of a compact surface M in E^m we have [2, I] (see also [5]),

$$(1.2) \quad \int_M \alpha^2 dV \geq 4\pi.$$

The equality holds when and only when M is an ordinary 2-sphere in an affine 3-space. It is an interesting problem to improve inequality (1.2) for some special surfaces in E^m . In [2, III] the author obtains some results of this problem for surfaces in E^4 . In this paper we shall study this problem for pseudo-umbilical surfaces in E^m (for the definition of pseudo-umbilicity see §2). In particular, we shall prove the following:

THEOREM 1. *Let M be a compact pseudo-umbilical surface in E^m with non-negative Gauss curvature. If we have*

$$(1.3) \quad \int_M \alpha^2 dV \leq (2 + \pi)\pi,$$

then M is homeomorphic to a 2-sphere.

THEOREM 2. *Let M be a compact flat pseudo-umbilical surface in E^m .*

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Then we have

$$(1.4) \quad \int_M \alpha^2 dV \geq 2\pi^2.$$

The equality sign holds if and only if M is a Clifford torus, i.e., M is the product surface of two plane circles with the same radius.

2. Preliminaries. Let $x: M \rightarrow E^m$ be an isometrical immersion of a surface M in an m -dimensional euclidean space E^m and let ∇ and ∇' be the covariant differentiations of M and E^m respectively. Let X and Y be two tangent vector fields on M . Then the second fundamental form h is given by

$$(2.1) \quad \nabla'_X Y = \nabla_X Y + h(X, Y).$$

It is well known that $h(X, Y)$ is a normal vector field on M and is symmetric on X and Y . Let ξ be a normal vector field on M ; we write

$$(2.2) \quad \nabla'_X \xi = -A_\xi(X) + D_X \xi,$$

where $-A_\xi(X)$ and $D_X \xi$ denote the tangential and normal components of $\nabla'_X \xi$. Then we have

$$(2.3) \quad \langle A_\xi(X), Y \rangle = \langle h(X, Y), \xi \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in E^m . A normal vector field ξ on M is said to be *parallel* (in the normal bundle) if $D\xi = 0$. The *mean curvature vector* H is defined by

$$(2.4) \quad H = \frac{1}{2} \text{trace } h.$$

The length of H , denoted by α , is called the *mean curvature* of M . If the mean curvature vector H is nowhere zero and the second fundamental form h satisfies

$$(2.5) \quad \langle h(X, Y), H \rangle = \lambda \langle X, Y \rangle,$$

for all tangent vectors X, Y on M , then M is said to be *pseudo-umbilical*.

Let R and R^N be the curvature tensors associated with connections ∇ and D , i.e., R and R^N are given respectively by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$$

and

$$R^N(X, Y) = [D_X, D_Y] - D_{[X, Y]}.$$

For a surface M in R^m , if R vanishes identically, then M is said to be flat. If R^N vanishes identically, then M is said to have *flat normal connection*. Let e_1 and e_2 be orthonormal vector fields tangent to M . Then the Gauss curvature G of M is a well-defined intrinsic function on M given by

$$G = \langle R(e_1, e_2)e_2, e_1 \rangle.$$

The Gauss and Ricci equations are given respectively by

$$(2.6) \quad \langle R(X, Y)Z, W \rangle = \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle,$$

$$(2.7) \quad \langle R^N(X, Y)\xi, \eta \rangle = \langle h(A_\xi(Y), X), \eta \rangle - \langle h(A_\xi(X), Y), \eta \rangle,$$

where X, Y, Z, W are vector fields tangent to M and ξ, η are vector fields normal to M . For the second fundamental form h , we define the covariant derivative, denoted by $\bar{\nabla}_X$, to be

$$(2.8) \quad (\bar{\nabla}_X h)(Y, Z) = D_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

The Codazzi equation is given by

$$(2.9) \quad (\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z).$$

Let $e_1, e_2, \xi_3, \dots, \xi_m$ be orthonormal vector fields, defined along M , such that e_1, e_2 are tangent to M and $\xi_3, \dots, \xi_m$ are normal to M . For each $r = 3, \dots, m$, we simply denote A_{ξ_r} by A_r . Let

$$A_r = (h_{ij}^r)_{ij=1,2}.$$

With respect to the basis e_1, e_2 , we have $h_{12}^r = h_{21}^r$. From (2.4) and (2.6) we find

$$(2.10) \quad H = \frac{1}{2} \sum (h_{11}^r + h_{22}^r) \xi_r,$$

$$(2.11) \quad G = \sum (h_{11}^r h_{22}^r - h_{12}^r h_{12}^r).$$

3. Mean curvature of pseudo-umbilical surfaces with $R = R^N = 0$. In this section we shall prove the following results for later use.

THEOREM 3. *Let M be a flat pseudo-umbilical surface in E^m with flat normal connection. Then the mean curvature α satisfies the following Laplace's equation,*

$$\Delta \ln \alpha = 0,$$

where Δ denotes the Laplacian on M .

PROOF. Since the normal connection of M is flat, the equation of Ricci implies that

$$(3.1) \quad [A_r, A_s] = 0, \quad r, s = 3, \dots, m.$$

Now, let $\xi_3, \dots, \xi_m$ be chosen in the way that $H = \alpha \xi_3$. Then by the pseudo-umbilicity of M in E^m , we have

$$(3.2) \quad A_3 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad A_r = \begin{pmatrix} h'_{11} & h'_{12} \\ h'_{12} & -h'_{11} \end{pmatrix}, \quad r = 4, \dots, m.$$

By (3.1), we may choose orthonormal tangent vectors e_1, e_2 which diagonalize A_r simultaneously. With respect to such frame $e_1, e_2, \xi_3, \dots, \xi_m$, we have

$$(3.3) \quad A_3 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad A_r = \begin{pmatrix} h'_{11} & 0 \\ 0 & -h'_{11} \end{pmatrix}, \quad r = 4, \dots, m.$$

Now, by the flatness of M and (2.11), we have

$$(3.4) \quad \alpha^2 = \sum_{r=4}^m (h'_{11})^2.$$

For each $p \in M$, let N_p be the vector space consisting of all normal vectors of M in E^m at p which are perpendicular to the mean curvature vector H . On N_p we define a linear mapping into the set of all symmetric matrices of order 2 by $\rho(\xi) = A_\xi$. Let O_p denote the kernel of ρ . Then by (3.3) and (3.4) we see that $\dim O_p = m - 4$. Hence, we may choose a frame field $e_1, e_2, \tilde{\xi}_3, \tilde{\xi}_4, \dots, \tilde{\xi}_m$ such that, with respect to this frame, we have

$$(3.5) \quad A_3 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad A_4 = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \quad A_s = 0, s = 5, \dots, m.$$

Since the normal connection is flat, there exist, at least locally, orthonormal normal vector fields $\bar{\xi}_3, \dots, \bar{\xi}_m$ such that $\bar{\xi}_3, \dots, \bar{\xi}_m$ are parallel (see [1, p. 99]), i.e.,

$$(3.6) \quad D\bar{\xi}_3 = \dots = D\bar{\xi}_m = 0.$$

We put

$$\bar{\xi}_r = \sum_{s=3}^m a_{rs} \tilde{\xi}_s, \quad r = 3, \dots, m.$$

Then (a_{rs}) is an orthogonal matrix of order $m - 2$.

Since M is two dimensional and local study of M is sufficient, we may assume that M is covered by an isothermal coordinate (x, y) such that the metric on M has the form $ds^2 = E(dx^2 + dy^2)$. In the following, we shall denote the coordinate vector fields $\partial/\partial x$ and $\partial/\partial y$ by X_1 and X_2 respectively. We put

$$L = h(X_1, X_1), \quad M = h(X_1, X_2), \quad N = h(X_2, X_2)$$

and

$$\nabla_{X_j} X_i = \sum_{k=1}^2 \Gamma_{ji}^k X_k, \quad i, j = 1, 2.$$

Then we have

$$\Gamma_{11}^1 = \Gamma_{12}^2 = -\Gamma_{22}^1 = X_1 E/2E, \quad \Gamma_{22}^2 = \Gamma_{12}^1 = -\Gamma_{11}^2 = X_2 E/2E.$$

Therefore the Codazzi equation reduces to

$$D_{X_2} L - D_{X_1} M = (X_2 E)H,$$

$$D_{X_2} M - D_{X_1} N = -(X_1 E)H.$$

Since X_1 and X_2 are orthonormal, we may define a function $\theta = \theta(x, y)$ by

$$(3.7) \quad \begin{aligned} X_1 &\equiv \partial/\partial x = \cos \theta e_1 + \sin \theta e_2, \\ X_2 &\equiv \partial/\partial y = -\sin \theta e_1 + \cos \theta e_2. \end{aligned}$$

Then with respect to the frame field $X_1, X_2, \bar{\xi}_3, \dots, \bar{\xi}_m$, the second fundamental tensors are given by

$$A_r = \begin{pmatrix} \alpha(a_{r1} + a_{r2} \cos 2\theta) & -\alpha a_{r2} \sin 2\theta \\ -\alpha a_{r2} \sin 2\theta & \alpha(a_{r1} - a_{r2} \cos 2\theta) \end{pmatrix}.$$

Since M is flat, we may assume that $\bar{E} = 1$. Hence by (3.6) equations of Codazzi reduce to

$$(3.8) \quad \frac{\partial}{\partial y} [\alpha(a_{r1} + a_{r2} \cos 2\theta)] = -\frac{\partial}{\partial x} [\alpha a_{r2} \sin 2\theta],$$

$$(3.9) \quad -\frac{\partial}{\partial y} [\alpha a_{r2} \sin 2\theta] = \frac{\partial}{\partial x} [\alpha(a_{r1} - a_{r2} \cos 2\theta)].$$

Multiplying a_{r1} to (3.8) and summing over r , then, by the fact that $(a_{rs}) \in O(m-2)$, we find

$$(3.10) \quad \frac{\partial \ln \alpha}{\partial y} = \sum_{r=3}^m \left\{ \left(\frac{\partial a_{r1}}{\partial y} \right) a_{r2} \cos 2\theta + \left(\frac{\partial a_{r1}}{\partial x} \right) a_{r2} \sin 2\theta \right\}.$$

Similarly, multiplying a_{r1} to (3.9) and summing over r , we have

$$(3.11) \quad \frac{\partial \ln \alpha}{\partial x} = \sum_{r=3}^m \left\{ \left(\frac{\partial a_{r1}}{\partial y} \right) a_{r2} \sin 2\theta - \left(\frac{\partial a_{r1}}{\partial x} \right) a_{r2} \cos 2\theta \right\}.$$

Multiplying a_{r2} to (3.8) and summing over r , we find

$$\begin{aligned} \sum a_{r2} \left(\frac{\partial a_{r1}}{\partial y} \right) + \left(\frac{\partial \ln \alpha}{\partial x} \right) \sin 2\theta + \left(\frac{\partial \ln \alpha}{\partial y} \right) \cos 2\theta \\ = 2 \frac{\partial \theta}{\partial y} \sin 2\theta - 2 \frac{\partial \theta}{\partial x} \cos 2\theta. \end{aligned}$$

Hence, by substituting (3.10) and (3.11) into this equation, we find

$$(3.12) \quad \sum a_{r2} \frac{\partial a_{r1}}{\partial y} = \sin 2\theta \frac{\partial \theta}{\partial y} - \cos 2\theta \frac{\partial \theta}{\partial x}.$$

Similarly, by multiplying a_{r2} to (3.9), summing over r , and by using (3.10) and (3.11), we get

$$(3.13) \quad \sum a_{r2} \frac{\partial a_{r1}}{\partial x} = -\cos 2\theta \frac{\partial \theta}{\partial y} - \sin 2\theta \frac{\partial \theta}{\partial x}.$$

Substituting (3.12) and (3.13) into (3.10) and (3.11), we may find

$$(3.14) \quad \frac{\partial \ln \alpha}{\partial x} = \frac{\partial \theta}{\partial y}, \quad \frac{\partial \ln \alpha}{\partial y} = -\frac{\partial \theta}{\partial x}.$$

From this we get $(\partial^2/\partial x^2) \ln \alpha + (\partial^2/\partial y^2) \ln \alpha = 0$. Since $E = 1$, this implies that $\Delta \ln \alpha = 0$. Q.E.D.

It shall be remarked that for a pseudo-umbilical surface in E^4 , the normal connection is always flat.

As an application of Theorem 3, we have the following result concerning about the *value-distribution* of mean curvature α .

THEOREM 4. *Let M be a complete flat pseudo-umbilical surface in E^m with flat normal connection. Then we have either*

- (1) *the mean curvature α of M takes every value in $(0, \infty)$, or*
- (2) *the mean curvature α of M takes only one value in $(0, \infty)$.*

If case (2) holds, M is the product of two curves C_1 and C_2 where C_1 is a curve in E^n for some n , $1 < n < m$; and C_2 is a curve in E^{m-n} so that the first curvatures of C_1 and C_2 are equal.

PROOF. Since M is flat and complete, M is parabolic in the sense that there exists no nonconstant negative subharmonic function on M . Thus every subharmonic function on M which is bounded from above on M must be a constant function. By Theorem 3, $\ln \alpha$ is a continuous harmonic function and so a subharmonic and also a superharmonic function on M . Hence, if α does not take every value in $(0, \infty)$, then α must be constant. This proves the first part of the theorem.

Now, suppose that case (2) holds. Then α is a nonzero constant. From (3.5) we see that with respect to the frame field $e_1, e_2, \xi'_3, \xi'_4, \xi'_5, \dots, \xi'_m$, we have

$$(3.15) \quad A_3 = \begin{pmatrix} \sqrt{2}\alpha & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2}\alpha \end{pmatrix}, \quad A_5 = \dots = A_m,$$

where $\xi'_3 = \cos \theta \tilde{\xi}_3 + \sin \theta \tilde{\xi}_4$, $\xi'_4 = \sin \theta \tilde{\xi}_3 - \cos \theta \tilde{\xi}_4$, $\xi'_5 = \tilde{\xi}_5, \dots, \xi'_m = \tilde{\xi}_m$. From (3.15) and the equations of structure we may easily find that both the distributions $T_i = \{ae_i: a \in R\}$, $i = 1, 2$ are parallel. By the de Rham decomposition theorem we see that $M = C_1 \times C_2$ where C_1 (respectively, C_2) is the maximal integral manifold of T_1 (resp. T_2). Moreover, (3.15) implies that $h(e_1, e_2) = 0$. Hence, M is the product of C_1 and C_2 such that C_1 is a curve in an affine n -space E^n and C_2 is a curve in an affine $(m-n)$ -space E^{m-n} in E^m . Let h^i be the second fundamental form of C_i , $i = 1, 2$. Then, (3.15) implies $h^i(e_i, e_i) = \sqrt{2}\alpha\xi'_{i+2}$, $i = 1, 2$. Hence the first curvatures of C_1 and C_2 are equal. Q.E.D.

4. Proofs of Theorems 1 and 2. Let M be a compact pseudo-umbilical surface in E^m . Let $e_1, e_2, \xi_3, \dots, \xi_m$ be a local field of orthonormal frame defined along M such that e_1, e_2 are tangent to M and $\xi_3, \dots, \xi_m$ normal to M . For a unit normal vector ξ at $p \in M$, the Lipschitz-Killing curvature $K(p, \xi)$ is defined by $K(p, \xi) = \det A_\xi$. Let $\xi = \sum_{r=3}^m \cos \theta_r \xi_r$ and $A_r = (h^r_{ij})$. Then we have

$$K(p, \xi) = \left(\sum \cos \theta_r h^r_{11} \right) \left(\sum \cos \theta_s h^s_{22} \right) - \left(\sum \cos \theta_t h^t_{12} \right)^2.$$

The right-hand side of this equation is a quadratic form of $\cos \theta_3, \dots, \cos \theta_m$. Hence, by choosing a suitable local frame field $\bar{\xi}_3, \dots, \bar{\xi}_m$, we may write

$$(4.1) \quad K(p, \xi) = \sum \lambda_{r-2}(p) \cos^2 \theta_r,$$

where $\lambda_1, \dots, \lambda_{m-2}$ are continuous functions defined on M and satisfy the following relations: $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{m-2}$. From (2.11) we find

$$(4.2) \quad G = \lambda_1 + \dots + \lambda_{m-2}.$$

Since M is pseudo-umbilical, if we choose ξ_3 in the direction of the mean curvature vector, then (3.2) holds. From this we may easily see that $K(p, \xi)$ takes its maximal value at ξ_3 . Hence we have

$$(4.3) \quad \lambda_1 = \alpha^2, \quad \lambda_2 \leq 0, \dots, \lambda_{m-2} \leq 0.$$

Now, let S_p be the unit $(m-3)$ -sphere of all unit normal vectors at $p \in M$ and $d\sigma$ the volume element of S_p . Then the total absolute curvature $K^*(p)$ at p is given by

$$K^*(p) = \int_{S_p} |K(p, \xi)| d\sigma$$

and the total absolute curvature $TA(M)$ of M in E^m (in the sense of Chern-Lashof [3]) is given by $TA(M) = \int_M K^*(p) dV$. Let $H_i(M; F)$ be the i th homology group

of M over a field F and $\beta_i(M; F)$ the dimension of $H_i(M; F)$. Then we have [3]

$$(4.4) \quad TA(M) \geq c_{m-1} \beta(M),$$

where $\beta(M) = \max \{ \sum_{i=0}^2 \beta_i(M; F) : F \text{ fields} \}$ and c_{m-1} the area of unit $(m-1)$ -sphere.

From (4.2) and (4.3) we find

$$(4.5) \quad \begin{aligned} |K(p, \xi)| &= \left| G \cos^2 \theta_3 + \sum_{r=4}^m \lambda_{r-2} (\cos^2 \theta_r - \cos^2 \theta_3) \right| \\ &\leq |G| \cos^2 \theta_3 - \sum_{r=4}^m \lambda_{r-2} |\cos^2 \theta_r - \cos^2 \theta_3|. \end{aligned}$$

This implies that

$$K^*(p) \leq \frac{c_{m-1}}{2\pi} |G| - \sum_{r=4}^m \lambda_{r-2} \int_{S_p} |\cos^2 \theta_r - \cos^2 \theta_3| d\sigma.$$

On the other hand, since

$$\int_{S_p} |\cos^2 \theta_r - \cos^2 \theta_3| d\sigma = 2c_{m-1}/\pi^2,$$

(4.2), (4.3) and (4.5) imply

$$(4.6) \quad \alpha^2 \geq (\pi^2/2c_{m-1})K^*(p) + G(p) - (\pi/4)|G(p)|.$$

Case (1). $G = 0$. In this case, (4.4) and (4.6) imply

$$(4.7) \quad \int \alpha^2 dV \geq (\pi^2/2)\beta(M).$$

On the other hand, the flatness of M implies that M is either homeomorphic to a torus or a Klein bottle; in both cases, $\beta(M) = 4$. Hence (4.7) implies inequality (1.4). Now, if the equality of (1.4) holds, then we have

$$(4.8) \quad TA(M) = 4c_{m-1}.$$

Moreover, the inequality in (4.5) is actually an equality for all $(\theta_3, \dots, \theta_m)$ satisfying $\cos^2 \theta_3 + \cos^2 \theta_4 + \dots + \cos^2 \theta_m = 1$. Hence we have

$$(4.9) \quad \lambda_{m-2} = -\alpha^2, \quad \lambda_2 = \dots = \lambda_{m-3} = 0.$$

From this we see that $[A_r, A_s] = 0$ for all $r, s = 3, \dots, m$. By using equation (2.7) of Ricci, we see that the normal connection of M in E^m is flat. Thus, Theorem 4 implies that $M = C_1 \times C_2$, where C_1 and C_2 are two closed curves in E^n and E^{m-n} , respectively, with the same first curvature for some n , $1 < n < m$.

On the other hand by a result of Kuiper [4], we have

$$TA(M)/c_{m-1} = TA(C_1)TA(C_2)/c_{n-1}c_{m-n-1},$$

where $TA(C_i)$ is the total absolute curvature of C_i , $i = 1, 2$. Since $TA(C_1) \geq 2c_{n-1}$ and $TA(C_2) \geq 2c_{m-n-1}$, (4.8) implies $TA(C_1) = 2c_{n-1}$ and $TA(C_2) = 2c_{m-n-1}$. From these we know that both C_1 and C_2 are two plane circles with the same radius [3]. Thus M is a Clifford torus. This proves Theorem 2.

Case (2). $G \geq 0$ and $G \not\equiv 0$. In this case, M is either homeomorphic to a sphere or a real projective plane. Now, suppose that M is homeomorphic to a real projective plane. Then we have $\chi(M) = 1$ and $\beta(M) = 3$. Hence inequality (4.4) implies

$$\int_M \alpha^2 dV \geq \frac{3}{2}\pi^2 + \left(1 - \frac{\pi}{2}\right) \int_M G dV.$$

This, combining with Gauss-Bonnet's formula, gives

$$(4.10) \quad \int_M \alpha^2 dV \geq (2 + \pi)\pi.$$

In the equality of (4.10) holds, then the inequality in (4.5) is actually an equality for all $(\theta_3, \dots, \theta_m)$ satisfying $\cos^2 \theta_3 + \dots + \cos^2 \theta_m = 1$. Hence, we have either $\lambda_2 = \dots = \lambda_{m-2} = 0$ or $G = \lambda_3 = \dots = \lambda_{m-2} = 0$ pointwise. Now, let $U = \{p \in M: G(p) \neq 0\}$. Then U is a nonempty open subset of M . By the assumption of pseudo-umbilicity, U is totally umbilical in E^m . Hence the Gauss curvature G is positive constant on every component of U (see [1, p. 49]). From this we know that U is also a closed subset of M . Thus $U = M$ and M is an ordinary 2-sphere in E^m . This is a contradiction. This proves Theorem 1. Q.E.D.

REMARK 1. The real projective plane can be immersed in E^5 as a pseudo-umbilical surface with positive constant Gauss curvature and total mean curvature 6π .

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